MAC-CPTM Situations Project

Situation 50: Connecting Factoring with the Quadratic Formula

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Prompt

Mr. Jones suspected his students saw no direct connection between the work they had done on factoring quadratics and the quadratic formula.

Commentary

Any quadratic polynomial can be represented as a product of two binomials. Quadratic polynomials of the form $x^2 + bx + c$ can be represented as a product of the form $(x+m)(x+n)$, where m,n may have real or complex values. This situation deals specifically with those quadratic polynomials that can be represented as a product of the form $(x+m)(x+n)$ where $m,n \in \Re$. Partitioned area models emphasize the relationship between the sum $x^2 + bx + c$ and the product $(x+m)(x+n)$ in cases for which x is a positive real number. In each of the foci in this Situation, we will assume x is positive.

Mathematical Foci

Mathematical Focus 1

Partitioned area models can be used to represent $x^2 + bx + c = (x+m)(x+n)$ *where* $m,n \in \mathrm{Z}$.

Consider the quadratic polynomial $x^2 + 3x + 2$, and a partitioned area model (figure 1).

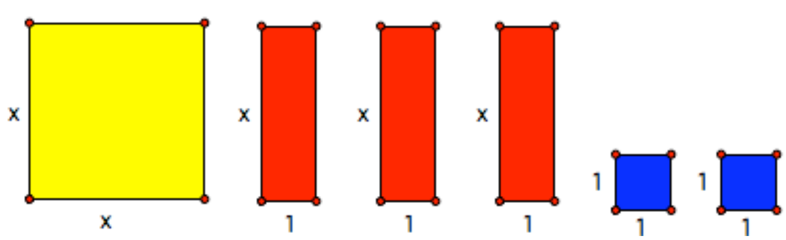

Figure 1

The process of factoring involves re-expressing a sum as a product. Hence, by composing a rectangle from the partitioned area model, the sum of the areas can be expressed as a product of the lengths of the sides of the rectangle (figure 2).

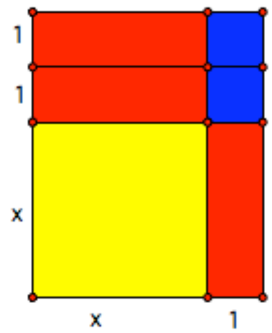

Figure 2

Since the area of the rectangle can be represented as the product of the lengths of the sides of the rectangle or as the sum of the partitioned areas, $x^2 + 3x + 2 = (x+1)(x+2)$.

Mathematical Focus 2

Partitioned area models can be used to represent $x^2 - a^2 = (x+a)(x-a)$ *where* $a \in \mathrm{Z}$.

Consider the quadratic polynomial $x^2 - 9$ and a partitioned area model (figure 3).

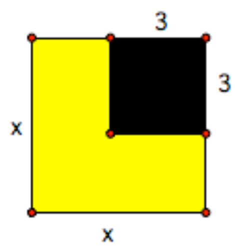

Figure 3

To factor the quadratic polynomial $x^2 - 9$, compose a rectangle from the partitioned area model (figure 4).

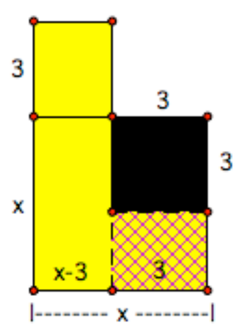

Figure 4

Since the area of the rectangle can be represented as the product of the lengths of the sides of the rectangle or as the sum of the partitioned areas, $x^2 - 9 = (x+3)(x-3)$.

Mathematical Focus 3

Partitioned area models can be used to represent the technique of completing the square for quadratic polynomials of the form $x^2 + bx$.

Now consider quadratic polynomials of the form $x^2 + bx$ such that a square can be composed from a partitioned area model representing $x^2 + bx + c$.

Consider the quadratic polynomial $x^2 + 6x$ and a partitioned area model (figure 5).

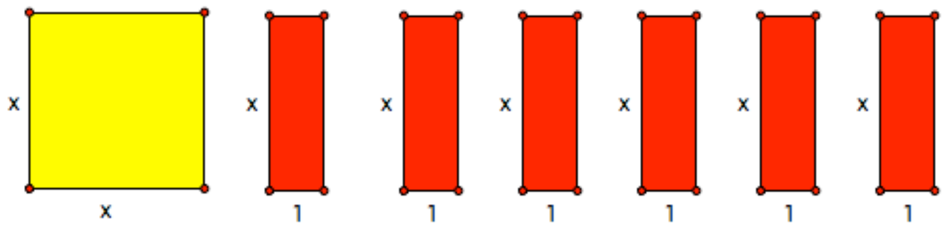

Figure 5

To compose a square, construct two adjacent rectangles: One with length x and width x+3 and another with length x and width 3 (figure 6).

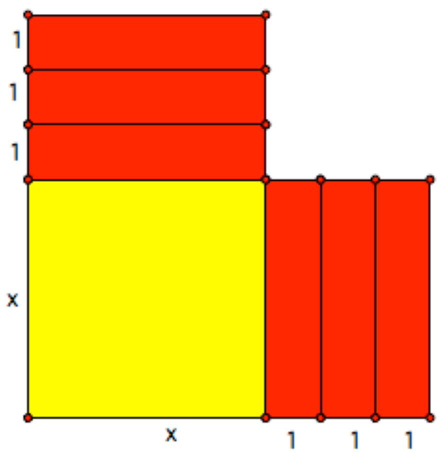

Figure 6

Composing a square requires adding 9 additional units of area in the form of a 3x3 square (figure 7).

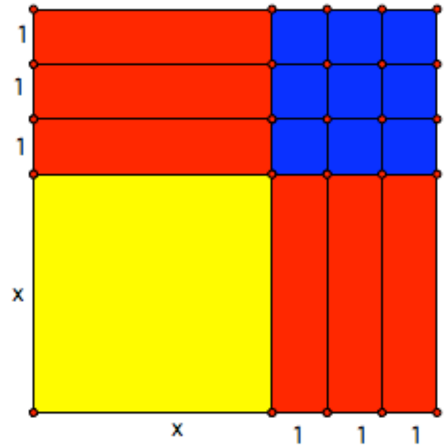

Figure 7

Hence, $x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2$.

Mathematical Focus 4

Partitioned area models can be used to represent $x^2 + bx + c = (x + m)(x + n)$ *where* $m, n \in \Re$.

Consider the quadratic polynomial $x^2 + 6x + 7$ and a partitioned area model (figure 8).

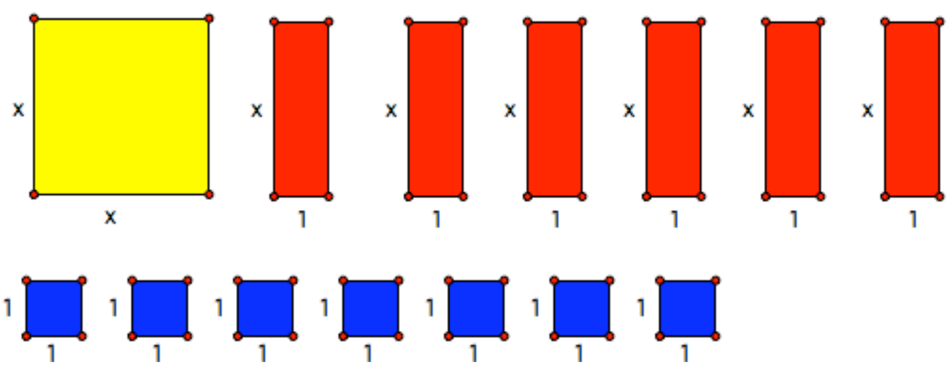

Figure 8

The partitioned area model is re-expressed in figure 9; however, a rectangle has not been composed.

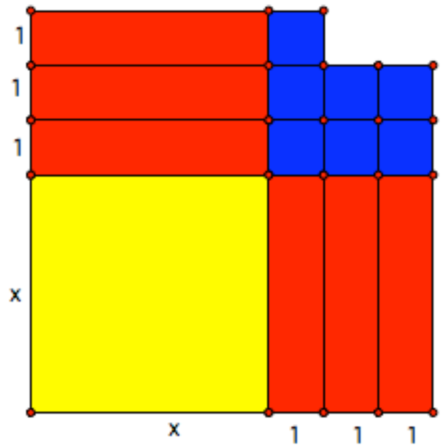

Figure 9

Since $x^2 + 6x + 7 = \left(x^2 + 6x + 9\right) - 2 = \left(x + 3\right)^2 - 2$, remove a square with two units of area from the area model representing $\left(x + 3\right)^2$ (figure 10).

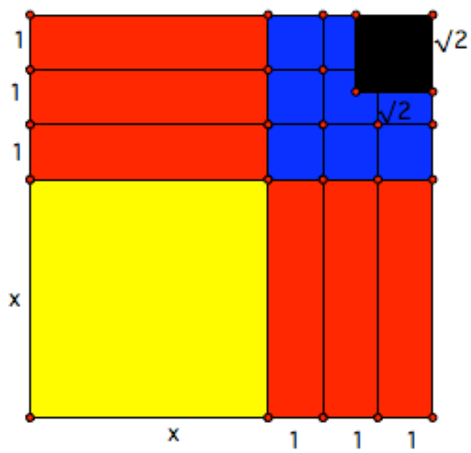

Figure 10

To factor the quadratic polynomial $x^2 + 6x + 7$, compose a rectangle from the partitioned area model (figure 11)

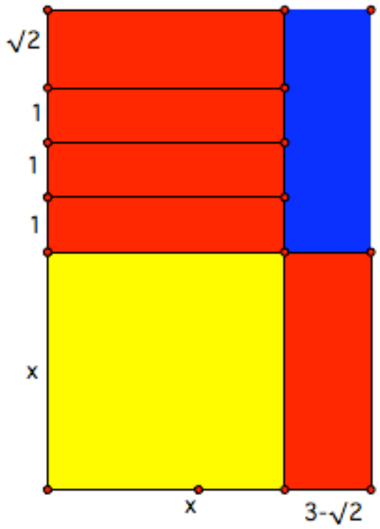

Figure 11

Since the area of the rectangle can be represented as the product of the lengths of the sides of the rectangle or as the sum of the partitioned areas,
$x^2 + 6x + 7 = \left(x + 3 + \sqrt{2}\right)\left(x + 3 - \sqrt{2}\right)$.

Mathematical Focus 5
Partitioned area models can be used to represent $x^2 + bx + c = (x + m)(x + n)$ *where* $m,n \in \Re$.

Consider the quadratic polynomial $x^2 + bx + c$ and a partitioned area model (figure 12).

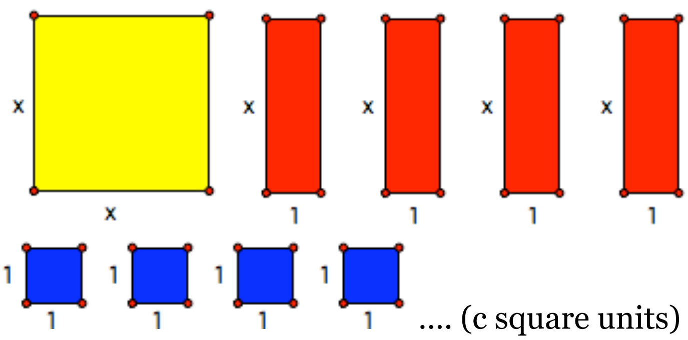

Figure 12

To factor the quadratic polynomial $x^2 + bx + c$, compose a rectangle from the partitioned area model (figures 13 and 14).

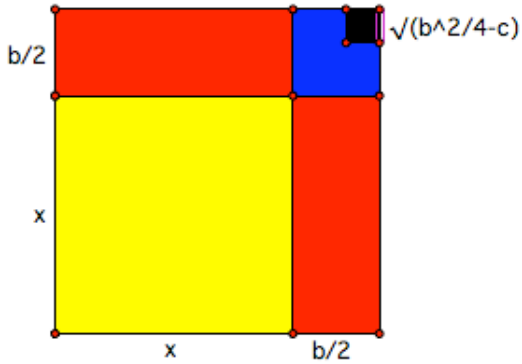

Figure 13

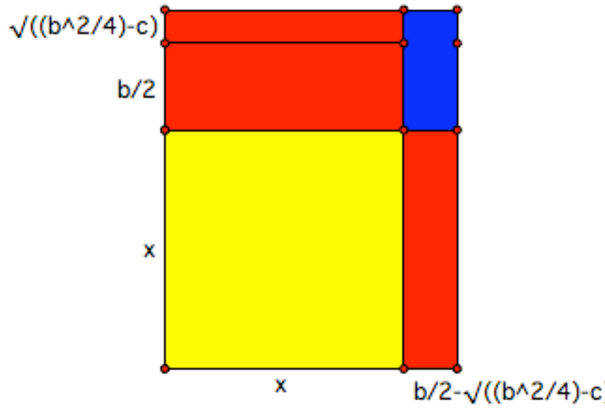

Figure 14

Since the area of the rectangle can be represented as the product of the lengths of the sides of the rectangle or as the sum of the partitioned areas,
$x^2 + bx + c = (x + \frac{b}{2} + \sqrt{\frac{b^2}{4} - c})(x + \frac{b}{2} - \sqrt{\frac{b^2}{4} - c})$.

The quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ can be used to determine solutions to a quadratic equation of the form $ax^2 + bx + c = 0$. When $a = 1$, the quadratic equation becomes $x^2 + bx + c = 0$, and applying the quadratic formula gives solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2} = \dfrac{-b}{2} \pm \sqrt{\dfrac{b^2 - 4c}{4}} = \dfrac{-b}{2} \pm \sqrt{\dfrac{b^2}{4} - c}$. Using the zero product property, solutions to the equation $x^2 + bx + c = (x + \frac{b}{2} + \sqrt{\frac{b^2}{4} - c})(x + \frac{b}{2} - \sqrt{\frac{b^2}{4} - c}) = 0$ can be determined: $x + \frac{b}{2} + \sqrt{\frac{b^2}{4} - c} = 0$ and $x + \frac{b}{2} - \sqrt{\frac{b^2}{4} - c} = 0$, resulting in $x = -\frac{b}{2} - \sqrt{\frac{b^2}{4} - c}$ and $x = -\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}$.